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Equivalences between Pure Type Systems and Systems of Illative Combinatory Logic

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Abstract Pure Type Systems, PTSs, were introduced as a generalization of the type systems of Barendregt's lambda cube and were designed to provide a foundation for actual proof assistants which will verify proofs. Systems of illative combinatory logic or lambda calculus, ICLs, were introduced by Curry and Church as a foundation for logic and mathematics. In an earlier paper we considered two changes to the rules of the PTSs which made these rules more like ICL rules. This led to four kinds of PTSs. Most importantly PTSs are about statements of the form $M : A$, where M is a term and A a type. In ICLs there are no explicit types and the statements are terms. In this paper we show that for each of the four forms of PTS there is an equivalent form of ICL, sometimes if certain conditions hold.

1 Introduction

The similarity between rules of a generalized type theory (that of Martin-Löf [16]) and those of illative combinatory logic was first noted in Bunder [8]. When Pure Type Systems (PTSs), which encompassed many generalized type systems, were developed, the similarity of the PTS application, abstraction, and product rules, and rules of illative systems of combinatory logic or lambda calculus (ICLs) such as those of Bunder [3] and [11] and Aczel [1] was still apparent.

There were, however, many differences. The most important was that PTSs have judgments of the form

$$x_1 : A_1, \ldots, x_n : A_n \vdash M : B \tag{1}$$

where in each statement $N : C$, C and N are "pseudoterms". ICLs' judgments take the form

$$X_1, \ldots, X_n \vdash Y \tag{2}$$

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where $X_1, \ldots, X_n$ and Y are combinatory or λ-terms.

In a paper preliminary to this one (Bunder and Dekkers [12]), we aimed to overcome at least the minor differences that were part of the gap between PTSs and ICLs by developing variants of PTSs with ICL-like properties. One difference is that in PTSs the "context", $x_1 : A_1, \ldots, x_n : A_n$, in (1) must be a *sequence* whose statements are introduced by one of two rules. For ICLs, the context $X_1, \ldots, X_n$ in (2) is an *arbitrary set* of terms. Also in ICLs, terms can be replaced by terms that are β- or $\beta\eta$-equal to them; for PTSs such substitutions are restricted. We introduced "set based PTSs" (SPTSs) which allow sets of judgments as contexts and unrestricted substitution of β-equality.

Also in PTSs, the abstraction rule has a stronger restriction than normally in ICLs. In [12] we introduced "abstraction altered PTSs" (APTSs) with an ICL-like abstraction rule. Finally we introduced SAPTSs which encorporated both changes.

We then showed, under what conditions and for which PTSs, PTS judgments were equivalent to SPTS, APTS, and SAPTS judgments. These new PTS-variants have independent interest in that they show that for many PTSs many of the rules can be relaxed. These more flexible PTS-variants are also closer to their formulas as type interpretations.

In this paper we show that, for each PTS, APTS, SPTS, and SAPTS, there is a corresponding ICL. That corresponding to a SAPTS is closest to the ICL used in the foundations of mathematics by Church, Curry, and their followers. (For details see Curry, Hindley, and Seldin [15], Bunder [13], or some of the series of papers in this journal which included Bunder [3] and [4].)

In each case, we show under what conditions the PTSs, APTSs and so on are equivalent to their ICL counterparts. Such equivalences hold for most standard PTSs, for example, the Calculus of Constructions. It was surprising that it was possible to extract from a large number of ICL-judgments of the form (2), variables $x_1, \ldots, x_n$, types $A_1, \ldots, A_n, B$ and a term M to give equivalent PTS-judgments of the form (1).

The new ICL-systems are shown in the top face of the cube in Figure 1 on the next page. The equivalences between the type and illative systems are shown by the lines joining them. Restrictions to some equivalences, the PTS being normalizing (n), a β-equal version of a judgment being provable only (β), the contexts being "legal" (L) and the condition ($*$), valid for many, but not all PTSs, are shown. The numbers of the theorems proving the results are also indicated.

2 The Pure Type Systems

Given a class of variables $V = \{x, y, z, \ldots, x_1, x_2, \ldots\}$ and a class of constants $\mathcal{C} = \{c_1, c_2, \ldots\}$ we have the following definition.

Definition 2.1 The class of pseudoterms $\mathcal{T}$ is given by

$$\mathcal{T} = V|\mathcal{C}|(\Pi V : \mathcal{T}.\mathcal{T})|(\lambda V : \mathcal{T}.\mathcal{T})|\mathcal{T}\mathcal{T}.$$

If $x \in V$ and $t_1, t_2 \in \mathcal{T}$, $(\lambda x : t_1.t_2)$ is interpreted as the λ-abstraction of t_2 with respect to the variable x of type t_1 and $(\Pi x : t_1.t_2)$ is interpreted as the class (or type) of all generalized functions from t_1 to t_2, where t_2 may be dependent on the argument x of the function. In $(\Pi x : t_1.t_2)$, x is bound just as in $(\lambda x : t_1.t_2)$. FV(t) will denote the set of free variables of t.

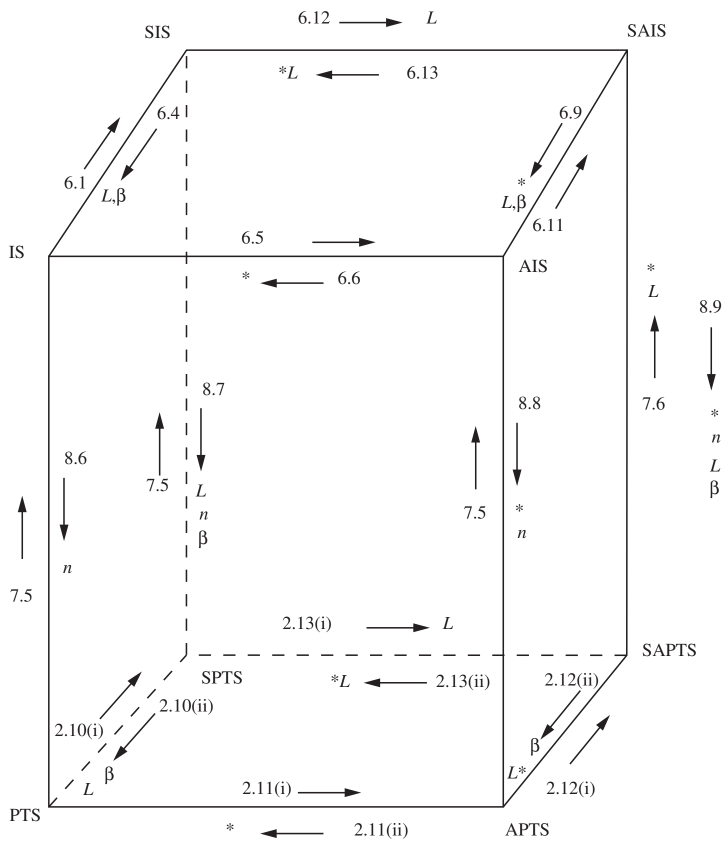

Figure 1

Definition 2.2

1. If M and A are *pseudoterms*, $M : A$ is a *statement*.
2. Γ is a *context* if it is a sequence of statements $\langle x_1 : A_1, \ldots, x_n : A_n \rangle$ where $x_1, \ldots, x_n \in V$. We will let $\mathrm{FV}(\Gamma)$ be the set of free variables of the pseudoterms in Γ.
3. If Γ is a context and M and A are pseudoterms then $\Gamma \vdash M : A$ is a *judgment*.

Definition 2.3 (Pure Type Systems, PTSs)

1. The *specification* of a PTS consists of a triple $\mathbf{S} = (\mathcal{S}, \mathcal{A}, \mathcal{R})$ where $\mathcal{S}$ is a subclass of $\mathcal{C}$ called the *sorts*, $\mathcal{A}$ is a class of statements of the form $(c : s)$, and $\mathcal{R}$ is a subclass of $\mathcal{S} \times \mathcal{S} \times \mathcal{S}$.
2. A *Pure Type System* (PTS) $\lambda\mathbf{S} = \lambda(\mathcal{S}, \mathcal{A}, \mathcal{R})$ determined by the specification $\mathbf{S} = (\mathcal{S}, \mathcal{A}, \mathcal{R})$ is defined as follows. Statements and contexts are as in Definition 2.2. The notion of type derivation, written as $\Gamma \vdash^{\lambda S} M : A$ (or just $\Gamma \vdash M : A$) is defined by the following postulates.

2.1 The PTS postulates

(axioms)	$\langle\,\rangle \vdash c : s$	where $c : s \in \mathcal{A}$
(start rule)	$\dfrac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}$	where $s \in \mathcal{S}$ and $x \notin FV(\Gamma)$
(weakening rule)	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma, x : B \vdash M : A}$	where $x \notin FV(\Gamma)$
(product rule)	$\dfrac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\Pi x : A.B) : s_3}$	where $(s_1, s_2, s_3) \in \mathcal{R}$
(abstraction rule)	$\dfrac{\Gamma, x : A \vdash M : B \quad \Gamma \vdash (\Pi x : A.B) : s}{\Gamma \vdash (\lambda x : A.M) : (\Pi x : A.B)}$	$s \in \mathcal{S}$
(application rule)	$\dfrac{\Gamma \vdash M : (\Pi x : A.B) \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B[x := N]}$	
(conversion rule)	$\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s \quad A =_\beta B}{\Gamma \vdash M : B}$	$s \in \mathcal{S}$

A pseudoterm A is *legal* in a PTS if, for some Γ and B, $\Gamma \vdash A : B$ or $\Gamma \vdash B : A$ in that PTS.

2.2 The SPTS postulates For an SPTS the above axioms and the start, weakening, and conversion rules are replaced by

(axioms)	$\Delta \vdash_{\mathrm{S}} c : s$	if $c : s \in \mathcal{A}$
(start)	$\Delta \vdash_{\mathrm{S}} M : A$	if $M : A \in \Delta$
(conversion)	$\dfrac{\Delta \vdash_{\mathrm{S}} M : A \quad \Delta =_\beta \Delta' \quad M =_\beta N \quad A =_\beta B}{\Delta' \vdash_{\mathrm{S}} N : B}$	

where Δ is an arbitrary set of statements $P : C$, rather than a sequence of statements $x : C$ formed using the start and weakening rules.

The remaining SPTS postulates are those of PTSs with $\vdash_{\mathrm{S}}$ for $\vdash$ and with each Γ (which we use for sequences) replaced by Δ (which we use for sets). The SPTS product and abstraction rules also require the restriction $x \notin FV(\Delta, A)$ which is derivable for PTSs.

2.3 The APTS postulates These are as for the PTS postulates except that $\vdash_{\mathrm{A}}$ is used for $\vdash$ and the abstraction rule is replaced by

(abstraction) $$\frac{\Gamma, x : A \vdash_{\mathrm{A}} M : B \quad \Gamma \vdash_{\mathrm{A}} A : s}{\Gamma \vdash_{\mathrm{A}} (\lambda x : A.M) : (\Pi x : A.B)}$$

where $s \in \mathcal{S}$ and $(+)\ \exists s_2, s_3 \left[(s, s_2, s_3) \in \mathcal{R}\ \&\ \forall D(B =_\beta D \in \mathcal{C} \Rightarrow (D : s_2) \in \mathcal{A})\right]$.

Note that this varies slightly from the $(+)$ in [12] which is

$$(+)^o \; \exists s_2, s_3\,[(s, s_2, s_3) \in \mathcal{R} \;\&\; (B \in \mathcal{C} \Rightarrow (B : s_2) \in \mathcal{A}]\,.$$

We will denote this "old" system, with $(+)^o$, by A^oPTS. Note also that $\Gamma \vdash_{\mathrm{A}} A : s$ is actually derivable whenever $\Gamma, x : A \vdash_{\mathrm{A}} M : B$ is, but we will retain it here as it is required for SAPTSs.

2.4 The SAPTS postulates These use $\vdash_{\mathrm{SA}}$ and have the alterations of the SPTSs and of the APTSs. We summarize below the results from [12] with minor variations due to the change from $(+)^o$ to $(+)$. We first need some definitions.

Definition 2.4 If Γ is the context $\langle x_1 : A_1, \ldots, x_n : A_n\rangle$, $S(\Gamma)$ is the set $\{x_1 : A_1, \ldots, x_n : A_n\}$.

Definition 2.5 A set Δ is S-legal in an SPTS if $\Delta =_\beta \{x_1 : A_1, \ldots, x_n : A_n\}$ and

(i) $(\forall i, j)[1 \le i < j \le n \Rightarrow x_i \ne x_j]$;
(ii) $(\forall i)\big[1 \le i \le n \Rightarrow (\exists s_i \in \mathcal{S})[x_1 : A_1, \ldots, x_{i-1} : A_{i-1} \vdash_{\mathrm{S}} A_i : s_i]\big]$;
(iii) $(\forall i)\,[1 \le i \le n \Rightarrow x_i, \ldots, x_n \notin FV(A_i)]$.

Definition 2.6 A context Γ is A-legal if for some M and B, $\Gamma \vdash_{\mathrm{A}} M : B$.

Definition 2.7 A set Δ is SA-legal if (i), (ii), and (iii) of Definition 2.5 hold with $\vdash_{\mathrm{SA}}$ for $\vdash_{\mathrm{S}}$.

Definition 2.8

(i) $\mathcal{S}_{\mathrm{I}}$ is the set generated by
  (a) $c : s \in \mathcal{A} \Rightarrow s \in \mathcal{S}_{\mathrm{I}}$,
  (b) $s : s' \in \mathcal{A} \Rightarrow s \in \mathcal{S}_{\mathrm{I}}$,
  (c) $s_1, s_2 \in \mathcal{S}_{\mathrm{I}} \;\&\; (s_1, s_2, s) \in \mathcal{R} \Rightarrow s \in \mathcal{S}_{\mathrm{I}}$;
(ii) $\mathcal{S}_1 = \{s_1 \in \mathcal{S}_{\mathrm{I}} | \exists s_2, s_3[(s_1, s_2, s_3) \in \mathcal{R}]\}$;
(iii) $\mathcal{S}_3 = \{s_3 | \exists s_1, s_2 \in \mathcal{S}_{\mathrm{I}}[(s_1, s_2, s_3) \in \mathcal{R}]\}$.

The definition of $\mathcal{S}_{\mathrm{I}}$ above varies from, but is equivalent to, that in [12]. This we show in Theorem 4.27.

Condition 2.9 (∗) The condition $(*)$ is defined as

$$\forall s_1 \in \mathcal{S}_1 \forall s_2 \in \mathcal{S}\,[((\exists s \in \mathcal{S})s_2 : s \in \mathcal{A} \vee s_2 \in \mathcal{S}_3) \Rightarrow \;\; \exists s_3 (s_1, s_2, s_3) \in \mathcal{R}]\,.$$

Theorem 2.10 *For any* PTS *and* SPTS *with the same specification,*

(i) $\Gamma \vdash M : A \;\Rightarrow\; S(\Gamma)$ *is* S-*legal* $\;\&\; S(\Gamma) \vdash_{\mathrm{S}} M : A$;
(ii) $\Delta \vdash_{\mathrm{S}} M : A \;\&\; \Delta$ *is* S-*legal* $\Rightarrow \exists \Gamma,$
  $M', A'[\Delta =_\beta S(\Gamma) \;\&\; M =_\beta M' \;\&\; A =_\beta A' \;\&\; \Gamma \vdash M' : A']$.

Theorem 2.11 *For any* PTS *and* APTS *with the same specification,*

(i) $\Gamma \vdash P : C \Rightarrow \Gamma \vdash_{\mathrm{A}} P : C$;
(ii) *if* $(*)$ *holds, then* $\Gamma \vdash_{\mathrm{A}} P : C \Rightarrow \Gamma \vdash P : C$.

Theorem 2.12 *For any* APTS *and* SAPTS *with the same specification,*

(i) $\Gamma \vdash_{\mathrm{A}} M : A \Rightarrow S(\Gamma)$ *is* SA-*legal* $\&\ S(\Gamma) \vdash_{\mathrm{SA}} M : A$*;*
(ii) *if* $(*)$ *holds,* Δ *is* SA-*legal* $\&\ \Delta \vdash_{\mathrm{SA}} M : A \Rightarrow \exists \Gamma,$
$M', A' \big[\Delta =_\beta S(\Gamma)\ \&\ M =_\beta M'\ \&\ A =_\beta A'\ \&\ \Gamma \vdash_{\mathrm{A}} M' : A'\big]$.

Theorem 2.13 *For any* SPTS *and* SAPTS *with the same specification,*

(i) $\Delta \vdash_{\mathrm{S}} M : A\ \&\ \Delta$ *is* S-*legal* $\Rightarrow \Delta \vdash_{\mathrm{SA}} M : A\ \&\ \Delta$ *is* SA-*legal;*
(ii) *if* $(*)$ *holds,* $\Delta \vdash_{\mathrm{SA}} M : A\ \&\ \Delta$ *is* SA-*legal* $\Rightarrow \Delta \vdash_{\mathrm{S}} M : A\ \&\ \Delta$ *is* S-*legal.*

2.5 Comments Theorems 2.10, 2.11, 2.12(ii), and 2.13(ii) and their proofs are the same as, or only slight variations of, due to the change from $(+^\circ)$ to $(+)$, Theorems 5.4, 8.1, and 8.3, 9.6(ii), and 9.8($\Leftarrow$) in [12]. The proof of Theorem 2.12(i) is by a simple induction on the derivation of $\Gamma \vdash_{\mathrm{A}} M : A$. Theorem 2.13(i) follows from Theorems 2.10(ii), 2.11(i), 2.12(i), and conversion.

In order to illustrate the kinds of proofs required we present, in Section 5 below, the proofs of Theorems 2.11(ii) and 2.12(i). Note that in Theorems 2.12(i) and 2.13(i) condition $(*)$ is not needed; this generalizes Theorems 9.6(i) and 9.8 ($\Rightarrow$) in [12].

To relate APTSs to A^oPTSs we have the following theorem.

Theorem 2.14 *For any* APTS *and* A^oPTS *with the same specification,*

(i) $\Gamma \vdash_{\mathrm{A}} M : B \Rightarrow \Gamma \vdash_{A^o} M : B$*;*
(ii) *if* $(*)$ *holds,* $\Gamma \vdash_{A^o} M : B \Rightarrow \Gamma \vdash_{\mathrm{A}} M : B$.

Proof (i) This is obvious as the systems are identical except that abstraction for A^oPTS has a weaker version of $(+)$.

(ii) If $\Gamma \vdash_{A^o} M : B$ and $(*)$ hold, then by Theorem 8.5 of [12] (the A^oPTS version of Theorem 2.11(ii)) we have $\Gamma \vdash M : B$ and by Theorem 2.11(i) we have $\Gamma \vdash_{\mathrm{A}} M : B$. □

3 Illative Systems

For each PTS, SPTS, APTS, and SAPTS that we will set up a system of illative lambda calculus, combinatory logic could easily have been used instead. We will refer to all of these illative systems as ICLs.

Definition 3.1 The class of pseudoterms T is given by

$$T = V \mid \mathcal{C} \mid GT(\lambda V.T) \mid \lambda V.T \mid TT$$

where V and $\mathcal{C}$ are as in Section 2. G is an illative constant that corresponds to Π; it is related (see Notation 3.6) to the restricted generality used by Church and Curry.

Each ICL will have a specification $(\mathcal{S}, [\mathcal{A}], \mathcal{R})$ where $\mathcal{S} \subseteq \mathcal{C}$ is a set of sorts and $\mathcal{R}$ a set of triples of sorts, as for PTSs. $[\mathcal{A}]$ is the set of axioms of the form sc, where $c : s$ is an axiom of the PTS specified by $(\mathcal{S}, \mathcal{A}, \mathcal{R})$.

3.1 The IS postulates

(axioms) $\qquad \vdash_{\mathrm{I}} X \qquad$ where $X \in [\mathcal{A}]$

(start rule) $\qquad \dfrac{\Gamma \vdash_{\mathrm{I}} sX}{\Gamma, Xx \vdash_{\mathrm{I}} Xx} \qquad$ where $s \in \mathcal{S}$ and $x \notin FV(\Gamma, X)$

(weakening rule) $\qquad \dfrac{\Gamma \vdash_{\mathrm{I}} X \quad \Gamma \vdash_{\mathrm{I}} sY}{\Gamma, Yx \vdash_{\mathrm{I}} X} \qquad$ where $s \in \mathcal{S}$ and $x \notin FV(\Gamma, Y)$

(product rule) $\qquad \dfrac{\Gamma \vdash_{\mathrm{I}} s_1 X \quad \Gamma, Xx \vdash_{\mathrm{I}} s_2 Y}{\Gamma \vdash_{\mathrm{I}} s_3(GX(\lambda x.Y))} \qquad$ where $(s_1, s_2, s_3) \in \mathcal{R}$

(abstraction rule) $\qquad \dfrac{\Gamma, Xx \vdash_{\mathrm{I}} YZ \quad \Gamma \vdash_{\mathrm{I}} s(GX(\lambda x.Y))}{\Gamma \vdash_{\mathrm{I}} GX\,(\lambda x.Y)(\lambda x.Z)} \qquad$ where $s \in \mathcal{S}$

(application rule) $\qquad \dfrac{\Gamma \vdash_{\mathrm{I}} GX(\lambda x.Y)Z \quad \Gamma \vdash_{\mathrm{I}} XU}{\Gamma \vdash_{\mathrm{I}} (Y[x := U])(ZU)}$

(conversion rule) $\qquad \dfrac{\Gamma \vdash_{\mathrm{I}} XY \quad \Gamma \vdash_{\mathrm{I}} sZ \quad X =_\beta Z}{\Gamma \vdash_{\mathrm{I}} ZY} \qquad$ where $s \in \mathcal{S}$

Notation 3.2 Systems such as these were called *separated systems* in Curry, Hindley, and Seldin [15]. Most ICLs in the literature are not separated as they have the SIS and SAIS (conversion) rule below (possibly with $\beta\eta$-equality).

3.2 The SIS postulates These are as above with $\vdash_{\mathrm{SI}}$ for $\vdash_{\mathrm{I}}$ and Δ for Γ, except that (axioms) and the (start), (weakening), and (conversion) rules are replaced by

(axioms) $\qquad \Delta \vdash_{\mathrm{SI}} X \qquad$ if $X \in [\mathcal{A}]$

(start) $\qquad \Delta \vdash_{\mathrm{SI}} X \qquad$ if $X \in \Delta$

(conversion) $\qquad \dfrac{\Delta \vdash_{\mathrm{SI}} X \quad \Delta =_\beta \Delta' \quad X =_\beta Y}{\Delta' \vdash_{\mathrm{SI}} Y}.$

The product and abstraction rules require the condition $x \notin FV(\Delta, X)$. In all the postulates, Δ is an arbitrary set of terms.

3.3 The AIS postulates These are the IS-postulates with $\vdash_{\mathrm{AI}}$ for $\vdash_{\mathrm{I}}$ and (abstraction) replaced by

(abstraction) $\qquad \dfrac{\Gamma, Xx \vdash_{\mathrm{AI}} YZ \quad \Gamma \vdash_{\mathrm{AI}} sX}{\Gamma \vdash_{\mathrm{AI}} GX(\lambda x.Y)(\lambda x.Z)}$

where

$$(+) \quad \exists s_2, s_3 \left[(s, s_2, s_3) \in \mathcal{R} \;\&\; \forall U (Y =_\beta U \in \mathcal{C} \Rightarrow s_2 U \in [\mathcal{A}])\right].$$

($\Gamma \vdash_{\mathrm{AI}} sX$ is actually derivable if $\Gamma, Xx \vdash_{\mathrm{AI}} YZ$ is, but we retain it in the rule as it is required for SAISs.)

3.4 The SAIS postulates These have $\vdash_{\mathrm{SAI}}$ for $\vdash$ and include the changed postulates from the SISs and AISs, where now in the abstraction rule Y is not β-equal to an abstract. β-equality is, in general, not decidable unless the terms involved have normal form. The normalization property, which says that all pseudoterms in a valid judgment have normal form, is one of our conditions for the equivalence of SAISs and PTSs (see Theorem 9.2).

Definition 3.3 A context Γ is (A)I-legal if for some X, $\Gamma \vdash_{(\mathrm{A})\mathrm{I}} X$.

Lemma 3.4 *A context Γ is* (A)I-legal $\Leftrightarrow$ $\Gamma \equiv \langle X_1x_1, \ldots, X_nx_n \rangle$ *for some terms $X_1, \ldots, X_n$ and variables $x_1, \ldots, x_n$, where*

(i) $\forall i, j[1 \leq i < j \leq n \Rightarrow x_i \neq x_j]$;
(ii) $\forall i[1 \leq i \leq n \Rightarrow \exists s_i \in \mathcal{S}[X_1x_1, \ldots, X_{i-1}x_{i-1} \vdash_{(\mathrm{A})\mathrm{I}} s_iX_i]]$;
(iii) $\forall i[1 \leq i \leq n \Rightarrow x_i, \ldots, x_n \notin FV(X_i)]$.

Proof ($\Rightarrow$) By induction on the derivation of $\Gamma \vdash_{(\mathrm{A})\mathrm{I}} X$ in Definition 3.3.

($\Leftarrow$) By the start rule from (i) and (ii) for $i = n$. □

Definition 3.5 A set of statements Δ is said to be S(A)I-legal, in an S(A)IS, if $\Delta =_\beta \{X_1x_1, \ldots, X_nx_n\}$ and (i), (ii), and (iii) of Lemma 3.4 hold with $\vdash_{\mathrm{S(A)I}}$ for $\vdash_{(\mathrm{A})\mathrm{I}}$.

Notation 3.6 Illative systems were first set up by Church and Curry using the symbol Curry denoted by Ξ, instead of the G used above.

$$G = \lambda xyz.\Xi x(\lambda u.(yu)(zu)).$$

The SAIS application rule follows, using this definition, from Curry's (and Church's) Ξ-elimination rule:

$$\frac{\Delta \vdash \Xi XY \quad \Delta \vdash XU}{\Delta \vdash YU}.$$

Their Ξ introduction rules,

Curry: $$\frac{\Delta, Xu \vdash Yu \quad u \notin \mathrm{FV}(\Delta, XY)}{\Delta \vdash \Xi XY}$$

Church: $$\frac{\Delta, Xu \vdash Yu \quad \Delta \vdash XV \quad u \notin \mathrm{FV}(\Delta, XY)}{\Delta \vdash \Xi XY}$$

led to inconsistency.

The SAIS abstraction rule follows from the Ξ introduction rule of Bunder [11] and [3],

$$\frac{\Delta, Xu \vdash Yu \quad \Delta \vdash \mathbf{L}X \quad u \notin \mathrm{FV}(\Delta, XY)}{\Delta \vdash \Xi XY}$$

with $\mathbf{L} \equiv s$.

Notation 3.7 $\exists s \in \mathcal{S}$ will often be abbreviated to $\exists s$ and $\forall s \in \mathcal{S}$ to $\forall s$.

4 Lemmas and Definitions for PTSs, ISs, APTSs, and AISs

In Lemmas 4.1–4.11 we quote several well-known lemmas for PTSs from Barendregt [2] and others from [12], all without proofs.

Lemma 4.1 (Free Variable Lemma) *Let $x_1 : A_1, \ldots, x_n : A_n \vdash B : C$. Then*

(i) *the $x_1, \ldots, x_n$ are all distinct;*

(ii) $FV(B), FV(C) \subseteq \{x_1, \ldots, x_n\}$;
(iii) $FV(A_i) \subseteq \{x_1, \ldots, x_{i-1}\}$ *for* $1 \leq i \leq n$.

Lemma 4.2 (Start Lemma) *If Γ is a legal context then*

(i) $(c : s) \in \mathcal{A} \Rightarrow \Gamma \vdash c : s$,
(ii) $(x : A) \in \Gamma \Rightarrow \Gamma \vdash x : A$.

Lemma 4.3 (Substitution Lemma) $\Gamma, x : A \vdash B : C \;\&\; \Gamma \vdash D : A \Rightarrow \Gamma \vdash B[x := D] : C[x := D]$.

Lemma 4.4 (Correctness of Contexts Lemma) *If $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ then for each i, $1 \leq i \leq n$, there is an $s_i \in \mathcal{S}$ such that the derivation of $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$ contains a derivation of $x_1 : A_1, \ldots, x_{i-1} : A_{i-1} \vdash A_i : s_i$.*

Lemma 4.5 (Thinning Lemma) *If Γ and Γ' are legal contexts and $\Gamma \subseteq \Gamma'$ then $\Gamma \vdash M : A \Rightarrow \Gamma' \vdash M : A$.*

Lemma 4.6 (Combining Contexts Lemma) *If Γ_1 and Γ_2 are legal contexts and $FV(\Gamma_1) \cap FV(\Gamma_2) = \varnothing$ then Γ_1, Γ_2 is a legal context.*

Lemma 4.7 (Sharpened Generation Lemma) *If $\Gamma \vdash P : B$ then*

(i) $P \equiv c \in \mathcal{C} \Rightarrow (c : B) \in \mathcal{A} \vee \exists B'[B =_\beta B' \;\&\; (c : B') \in \mathcal{A} \;\&\; \exists s\, [\Gamma \vdash B : s]]$;

(ii) $P \equiv x \in \mathcal{V} \Rightarrow (x : B) \in \Gamma \vee \exists B'[B =_\beta B' \;\&\; (x : B') \in \Gamma \;\&\; \exists s\, [\Gamma \vdash B : s]]$;

(iii) $P \equiv (\Pi x : A.C) \Rightarrow \exists s_1, s_2, s_3[\Gamma \vdash A : s_1 \;\&\; \Gamma, x : A \vdash C : s_2$
$\;\&\; (s_1, s_2, s_3) \in \mathcal{R} \;\&\; (B \equiv s_3 \vee (B =_\beta s_3 \;\&\; (\exists s[\Gamma \vdash B : s])]$;

(iv) $P \equiv (\lambda x : A.M) \Rightarrow \exists C, s_3[\Gamma \vdash (\Pi x : A.C) : s_3 \;\&\; \Gamma, x : A \vdash M : C \;\&$
$(B \equiv \Pi x : A.C \vee (B =_\beta \Pi x : A.C \;\&\; \exists s[\Gamma \vdash B : s]))]$;

(v) $P \equiv MN \Rightarrow \exists A, C[\Gamma \vdash M : (\Pi x : A.C) \;\&\; \Gamma \vdash N : A \;\&$
$(B \equiv C[x := N] \vee (B =_\beta C[x := N] \;\&\; \exists s[\Gamma \vdash B : s]))]$.

and in each case the deductions without an explicit $x : A$ in the context are shorter than that of $\Gamma \vdash P : B$.

Lemma 4.8 (Correctness of Types Lemma)

$$\Gamma \vdash M : A \Rightarrow \exists s[A \equiv s \vee \Gamma \vdash A : s].$$

Lemma 4.9 (Subject Reduction Lemma)

$$\Gamma \vdash M : A, \Gamma \twoheadrightarrow_\beta \Gamma', M \twoheadrightarrow_\beta M', A \twoheadrightarrow_\beta A' \Rightarrow \Gamma' \vdash M' : A'.$$

Lemma 4.10

$$\Gamma =_\beta \Gamma' \;\&\; \Gamma' \textit{ is legal } \;\&\; \Gamma \vdash M : A \Rightarrow \Gamma' \vdash M : A.$$

Lemma 4.11 *If $\Gamma \vdash M : A$ then at least one of*

(i) $A \in \mathcal{C}$,
(ii) $\exists s[\Gamma \vdash A : s \;\&\; (\exists s'[s : s' \in \mathcal{A}] \vee s \in \mathcal{S}_3)]$.

In Lemmas 4.12 – 4.22 we have similar lemmas for ISs with similar proofs.

Lemma 4.12 (Free Variable Lemma for ISs) *Let $X_1x_1, \ldots, X_nx_n \vdash_I Y$. Then*

(i) *the $x_1, \ldots, x_n$ are all distinct;*
(ii) $FV(Y) \subseteq \{x_1, \ldots, x_n\}$;
(iii) $FV(X_i) \subseteq \{x_1, \ldots, x_{i-1}\}$ *for $i \leq i \leq n$.*

Lemma 4.13 (Start Lemma for ISs) *If Γ is an I-legal context, then*

(i) $sc \in [\mathcal{A}] \Rightarrow \Gamma \vdash_{\mathrm{I}} sc$,
(ii) $Xx \in \Gamma \Rightarrow \Gamma \vdash_{\mathrm{I}} Xx$.

Lemma 4.14 (Substitution Lemma for ISs)

$$\Gamma, Xx \vdash_{\mathrm{I}} Y \ \& \ \Gamma \vdash_{\mathrm{I}} XZ \Rightarrow \Gamma \vdash_{\mathrm{I}} Y[x := Z].$$

Lemma 4.15 (Correctness of Contexts Lemma for ISs) *If $X_1x_1, \ldots, X_nx_n \vdash_I Y$ then for each i, $1 \leq i \leq n$, there is an $s_i \in \mathcal{S}$ such that the derivation of $X_1x_1, \ldots, X_nx_n \vdash_{\mathrm{I}} Y$ contains a derivation of $X_1x_1, \ldots, X_{i-1}x_{i-1} \vdash_{\mathrm{I}} s_iX_i$.*

Lemma 4.16 (Thinning Lemma for ISs) *If Γ and Γ' are I-legal contexts and $\Gamma \subseteq \Gamma'$ then $\Gamma \vdash_{\mathrm{I}} Y \Rightarrow \Gamma' \vdash_{\mathrm{I}} Y$.*

Lemma 4.17 (Combining Contexts Lemma for ISs) *If Γ_1 and Γ_2 are I-legal contexts and $FV(\Gamma_1) \cap FV(\Gamma_2) = \varnothing$ then Γ_1, Γ_2 is an I-legal context.*

Lemma 4.18 (Sharpened Generation Lemma for ISs) *If $\Gamma \vdash_{\mathrm{I}} YZ$, then*

(i) $Z \equiv c \in \mathcal{C} \Rightarrow YZ \in [\mathcal{A}] \vee \exists T[Y =_\beta T \ \& \ TZ \in [\mathcal{A}] \ \& \ \exists s[\Gamma \vdash_{\mathrm{I}} sY]]$;

(ii) $Z \equiv x \in \mathcal{V} \Rightarrow Yx \in \Gamma \vee \exists T[Y =_\beta T \ \& \ Tx \in \Gamma \ \& \ \exists s[\Gamma \vdash_{\mathrm{I}} sY]]$;

(iii) $Z \equiv GU(\lambda x.V) \Rightarrow \exists s_1, s_2, s_3[\Gamma \vdash_{\mathrm{I}} s_1U \ \& \ \Gamma, Ux \vdash_{\mathrm{I}} s_2V \ \&$
$(s_1, s_2, s_3) \in \mathcal{R} \ \& \ (Y \equiv s_3 \vee (Y =_\beta s_3 \ \& \ \exists s[\Gamma \vdash_{\mathrm{I}} sY]))]$;

(iv) $Z \equiv \lambda x.T \Rightarrow \exists U, V, s_3[\Gamma \vdash_{\mathrm{I}} s_3(GU(\lambda x.V)) \ \& \ \Gamma, Ux \vdash_{\mathrm{I}} VT \ \&$
$(Y \equiv GU(\lambda x.V) \ \vee (Y =_\beta GU(\lambda x.V) \ \& \ \exists s[\Gamma \vdash_{\mathrm{I}} sY]))]$;
(v) $Z \equiv RT \Rightarrow \exists U, V[\Gamma \vdash_{\mathrm{I}} GU(\lambda x.V)R \ \& \ \Gamma \vdash_{\mathrm{I}} UT \ \&$
$(Y \equiv V[x := T] \ \vee (Y =_\beta V[x := T] \ \& \ \exists s(\Gamma \vdash_{\mathrm{I}} sY)))]$

where the derivations without an explicit Ux in the context are shorter than that of $\Gamma \vdash_{\mathrm{I}} YZ$.

Lemma 4.19 (Correctness of Types Lemma for ISs) $\Gamma \vdash UV \Rightarrow \exists s[U \equiv s \vee \Gamma \vdash sU]$.

Lemma 4.20 (Subject Reduction Lemma for ISs) $\Gamma \vdash_{\mathrm{I}} UV, \Gamma \twoheadrightarrow_\beta \Gamma', U \twoheadrightarrow_\beta U'$, $V \twoheadrightarrow_\beta V' \ \Rightarrow \ \Gamma' \vdash_{\mathrm{I}} U'V'$.

Lemma 4.21 $\Gamma =_\beta \Gamma' \ \& \ \Gamma'$ *is legal* $\& \ \Gamma \vdash X \Rightarrow \Gamma' \vdash X$.

Lemma 4.22 *If $\Gamma \vdash UV$ then at least one of*

(i) $U \in \mathcal{C}$,
(ii) $\exists s \left[\Gamma \vdash sU \ \& \ (\exists s'[s's \in \mathcal{A}]] \text{ or } s \in \mathcal{S}_3)\right]$.

We give three extra lemmas.

Lemma 4.23 *If $\Gamma \vdash_{\mathrm{I}} X$, then $X \equiv UV$ for some U and V, where U is not β-equal to an abstract.*

Proof $X \equiv UV$ follows by induction on the derivation of $\Gamma \vdash_{\mathrm{I}} X$. By the Correctness of Types Lemma, the Generation Lemma, and Subject Reduction for ISs, U is not β-equal to an abstract. □

Remark 4.24 By Lemma 4.23 we see that the Subject Reduction Lemma for ISs can be strengthened to

$$\Gamma \vdash_{\mathrm{I}} X, \Gamma \twoheadrightarrow \Gamma', X \twoheadrightarrow_{\beta} X' \Rightarrow \Gamma' \vdash_{\mathrm{I}} X'.$$

Lemma 4.25 (Sharpened Generation Lemma for APTSs) *If $\Gamma \vdash_{\mathrm{A}} P : B$, then (i), (ii), (iii), and (v) of Lemma 4.7 hold with $\vdash$ replaced by $\vdash_{\mathrm{A}}$ and also,*

(iv) $P \equiv (\lambda x : A.M) \Rightarrow \exists C, s\,[\Gamma \vdash_{\mathrm{A}} C : s \;\&\; \Gamma, x : A \vdash_{\mathrm{A}} M : C]$
$\quad\& \; \big(B \equiv \Pi x : A.C \vee (B =_{\beta} \Pi x : A.C \;\&\; \exists s'[\Gamma \vdash_{\mathrm{A}} B : s'])\big)].$

Proof As for Lemma 4.7. □

Lemma 4.26 (Sharpened Generation Lemma for AISs) *If $\Gamma \vdash_{\mathrm{AI}} YZ$ then (i), (ii), (iii), and (v) of Lemma 4.18 hold with $\vdash_{\mathrm{I}}$ replaced by $\vdash_{\mathrm{AI}}$ and also,*

(iv) $Z \equiv \lambda x.T \Rightarrow \exists U, V, s\,[\Gamma \vdash_{\mathrm{AI}} sU \;\&\; \Gamma, Ux \vdash_{\mathrm{AI}} VT \;\&$
$\quad\big(Y \equiv GU(\lambda x.V) \vee (Y =_{\beta} GU(\lambda x.V) \;\&\; \exists s'[\Gamma \vdash_{\mathrm{AI}} s'Y])\big)].$

We now prove a theorem which is of importance for PTSs in general and which is also useful in later proofs.

Theorem 4.27

(i) *$\mathcal{S}_{\mathrm{I}}$ is the set of inhabited sorts, that is, those sorts s such that $\exists \Gamma, M[\Gamma \vdash M : s]$.*
(ii) *For every* PTS *with specification $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ there is a* PTS *with specification $(\mathcal{S}_{\mathrm{I}}, \mathcal{A}, \mathcal{R} \cap \mathcal{S}_{\mathrm{I}}^{3})$ that is equivalent in the sense that it has the same valid judgments.*

Proof We prove

$$\begin{aligned} & \exists \Gamma, M, A[s \text{ appears in the statement } \Gamma \vdash M : A] && (1)\\ \Rightarrow\;& s \in \mathcal{S}_{\mathrm{I}} && (2)\\ \Rightarrow\;& \exists \Gamma', N[\Gamma' \vdash N : s] && (3)\\ \Rightarrow\;& (1). \end{aligned}$$

(1) $\Rightarrow$ (2) By induction on the derivation of

$$\Gamma \vdash M : A. \quad (4)$$

If (4) is an axiom then $M : A$ is $c : s$ or $s : s'$, so (a) or (b) of Definition 2.8(i) holds. If (4) comes by a product rule with $A \equiv s$, (c) holds. In all other cases the result holds by the induction hypothesis.

(2) $\Rightarrow$ (3) By induction on the derivation of (2). If this is by Definition 2.8(i)(a), the result holds; if it is by (b) it holds by a start rule. If this is by (c) we have $s_1, s_2 \in \mathcal{S}_{\mathrm{I}}$ and $(s_1, s_2, s) \in \mathcal{R}$ and by the induction hypothesis

$$\Gamma_1 \vdash N_1 : s_1$$

$$\Gamma_2 \vdash N_2 : s_2$$

where we can assume $FV(\Gamma_1) \cap FV(\Gamma_2) = \varnothing$.

By Lemmas 4.5 and 4.6 we have

$$\Gamma_1, \Gamma_2, \vdash N_1 : s_1$$

$$\Gamma_1, \Gamma_2 \vdash N_2 : s_2,$$

and by weakening

$$\Gamma_1, \Gamma_2, x : N_1 \vdash N_2 : s_2$$

where $x \notin FV(\Gamma_1, \Gamma_2, N_1, N_2)$. A product rule now gives (3).

(3) $\Rightarrow$ (1) is obvious.

(2) $\Leftrightarrow$ (3) now establishes (i).

(1) $\Leftrightarrow$ (2) shows that if $s \in \mathcal{S} - \mathcal{S}_I$, it cannot appear in any valid judgment. Hence only the sorts in S_I and the triples in $\mathcal{R} \cap \mathcal{S}_I^3$ can be used to derive any valid judgment. This establishes (ii). □

Theorem 4.28 *Theorem 4.27 holds for* ISs *if* $\Gamma \vdash M : s$ *in (i) is replaced by* $\Gamma \vdash_I sM$.

Proof Analogous to that of Theorem 4.27. □

5 Proof of Theorems 2.11(ii) and 2.12(i)

Proof of Theorem 2.11(ii) For a PTS and an APTS with the same specification and such that $(*)$ holds, we prove

$$\Gamma \vdash_A P : C \Rightarrow \Gamma \vdash P : C$$

by induction on the derivation of $\Gamma \vdash_A P : C$. The only nontrivial case is where $\Gamma \vdash_A P : C$ is obtained by the abstraction rule from

$$\Gamma, x : A \vdash_A M : B \;\&\; \Gamma \vdash_A A : s$$

where $(+)$ holds for s and B, $P \equiv \lambda x : A.M$ and $C \equiv \Pi x : A.B$. By the induction hypothesis we have

$$\Gamma, x : A \vdash M : B \;\&\; \Gamma \vdash A : s.$$

We only need to show

$$\exists s_3 \in \mathcal{S}\, [\Gamma \vdash \Pi x : A.B : s_3].$$

Lemma 4.11 applied to $\Gamma, x : A \vdash M : B$ yields that we have at least one of

(i) $B \in \mathcal{C}$,

(ii) $\Gamma, x : A \vdash B : s_2$ & $(\exists s'[s_2 : s' \in \mathcal{A}]$ or $s_2 \in \mathcal{S}_3)$.

In case (i) we get from $(+)$

$$(\exists s_2, s_3)[(s, s_2, s_3) \in \mathcal{R} \;\&\; \Gamma, x : A \vdash B : s_2],$$

so $\Gamma \vdash \Pi x : A.B : s_3$.

In case (ii), we get from $(*)$, $\exists s_3 (s, s_2, s_3) \in \mathcal{R}$ (note that $s \in \mathcal{S}_1$ by $(+)$), so $\Gamma \vdash \Pi x : A.B : s_3$. □

Proof of Theorem 2.12(i) We want $\Gamma \vdash_A M : A \Rightarrow S(\Gamma)$ is SA-legal & $S(\Gamma) \vdash_{SA} M : A$. We let $\Gamma \equiv \langle x_1 : A_1, \ldots, x_n : A_n \rangle$ and proceed by induction on the derivation of

$$\Gamma \vdash_A M : A.$$

Case axiom Now $\Gamma \equiv \langle \, \rangle$, $S(\Gamma) = \varnothing$ and is SA-legal, and $S(\Gamma) \vdash_{SA} M : A$.

Case start $\Gamma \equiv \Gamma^-, x_n : A_n, M \equiv x_n, A \equiv A_n$, and $\Gamma \vdash_A M : A$ is obtained from $\Gamma^- \vdash_A A_n : s$. By the induction hypothesis we have $S(\Gamma^-)$ is SA-legal and $S(\Gamma^-) \vdash_{SA} A_n : s$. Also $x_n \neq x_i$ and $x_n \notin FV(A_i)$ for $1 \leq i < n$, so $S(\Gamma)$ is SA-legal. $M : A \in S(\Gamma)$, hence $S(\Gamma) \vdash_{SA} M : A$.

Case weakening $\Gamma \equiv \Gamma^-, x_n : A_n$, and (1) is obtained from $\Gamma^- \vdash_A M : A$, $\Gamma \vdash_A A_n : s$. We have as above that $S(\Gamma)$ is legal. By the induction hypothesis we have $S(\Gamma^-) \vdash_{SA} M : A$. One easily proves the Thinning Lemma for SAPTSs:

$$\Delta \vdash_{SA} N : B, \Delta \subseteq \Delta' \Rightarrow \Delta' \vdash_{SA} N : B.$$

From this we get $S(\Gamma) \vdash M : A$.

Other Cases If $\Gamma \vdash_A M : A$ is obtained by one of the other rules, we find by the induction hypothesis applied to one of the premises from which $\Gamma \vdash_A M : A$ is obtained that $S(\Gamma)$ is S-legal. In each case $S(\Gamma) \vdash_{SA} M : A$ follows when the induction hypothesis is applied to the premises. □

6 Relations between Illative Systems

In each theorem and lemma in this and later sections we assume that the systems used have the same specification.

Theorem 6.1

$$\Gamma \vdash_I X \Rightarrow S(\Gamma) \textit{ is } \text{SI-}\textit{legal and } S(\Gamma) \vdash_{SI} X.$$

Proof By an easy induction on the derivation of $\Gamma \vdash_I X$, similar to the proof of Theorem 2.10(i). □

We now will prove a sort of converse:

$$\Delta \vdash_{SI} X,\ \Delta \text{ SI-legal } \Rightarrow \exists \Gamma, Y[\Delta =_\beta S(\Gamma) \;\&\; X =_\beta Y \;\&\; \Gamma \vdash_I Y].$$

The proof of this statement is very similar to the proof of Theorem 5.4(ii) in [12]. We first prove two lemmas, similar to Lemmas 5.2 and 5.3 in that paper, with similar proofs.

Lemma 6.2 *If*

$$\Delta \vdash_{SI} X, \tag{1}$$

$\Delta =_\beta S(\Gamma)$ *where* Γ *is* I*-legal, then there exists* $Y =_\beta X$ *such that*

$$\Gamma \vdash_I Y. \tag{2}$$

Proof By induction on the derivation of (1).

Case axiom $X \in [\mathcal{A}]$. Now (2) follows by the Start Lemma for ISs.

Case start $X \in \Delta =_\beta S(\Gamma)$. Now $X =_\beta Y$ for some $Y \in \Gamma$ and by the Start Lemma for ISs we get $\Gamma \vdash_I Y$.

Case product $X \equiv s_3(GU(\lambda x.V))$, $(s_1, s_2, s_3) \in \mathcal{R}$, and (1) is obtained from

$$\Delta \vdash_{\mathrm{SI}} s_1 U, \text{ and} \tag{3}$$

$$\Delta, Ux \vdash_{\mathrm{SI}} s_2 V. \tag{4}$$

By the induction hypothesis for (3), Lemma 4.23, and Subject Reduction for ISs, we get for some $U' =_\beta U$,

$$\Gamma \vdash_{\mathrm{I}} s_1 U'. \tag{5}$$

So $\Gamma, U'x$ is I-legal. As $\Delta, Ux =_\beta S(\Gamma, U'x)$ we have by the induction hypothesis for (4), Lemma 4.23, and Subject Reduction

$$\Gamma, U'x \vdash_{\mathrm{I}} s_2 V' \tag{6}$$

where $V' =_\beta V$.

By the product rule we get from (5) and (6),

$$\Gamma \vdash s_3(GU'(\lambda x.V')).$$

This is (2) with $Y \equiv s_3(Gu'(\lambda x.V'))$.

Case abstraction $X \equiv GU(\lambda x.V)(\lambda x.W)$ and (1) is obtained from

$$\Delta, Ux \vdash_{\mathrm{SI}} VW, \text{ and} \tag{7}$$

$$\Delta \vdash_{\mathrm{SI}} s(GU(\lambda x.V)). \tag{8}$$

By the induction hypothesis applied to (8) and Subject Reduction for ISs we get

$$\Gamma \vdash_{\mathrm{I}} s(GU'(\lambda x.V'))$$

for some $U' =_\beta U$, $V' =_\beta V$, and hence by Lemma 4.18,

$$\Gamma \vdash_I s_1 U' \text{ and } \Gamma, U'x \vdash s_2 V'.$$

Hence, also by Lemma 4.18, U' and V', so also U and V, are not β-equal to abstracts.

From $\Gamma \vdash_{\mathrm{I}} s_1 U'$ we get that $\Gamma, U'x$ is I-legal. Hence we get by the induction hypothesis applied to (7),

$$\Gamma, U'x \vdash_{\mathrm{I}} V''W'' \text{ where } V'' =_\beta V, W'' =_\beta W.$$

By Church-Rosser and Subject Reduction finally we get

$$\Gamma, U_1 x \vdash_{\mathrm{I}} V_1 W_1, \Gamma \vdash_{\mathrm{I}} s(GU_1(\lambda x.V_1))$$

where $U_1 =_\beta U$, $V_1 =_\beta V$ and $W_1 =_\beta W$.

We conclude (2) by the abstraction rule for ISs.

Case application $X \equiv (V[x := R])(WR)$ and (1) is obtained from

$$\Delta \vdash_{\mathrm{SI}} GU(\lambda x.V)W, \text{ and} \tag{9}$$

$$\Delta \vdash_{\mathrm{SI}} UR. \tag{10}$$

Similar to the previous case, now also using Lemma 4.19, we get that U is not β-equal to an abstract. We get

$$\Gamma \vdash_{\mathrm{I}} GU'(\lambda x.V')W' \;\&\; \Gamma \vdash_{\mathrm{I}} U'R'$$

for some $U' =_\beta U$, $V' =_\beta V$, $W' =_\beta W$, and $R' =_\beta R$. Hence by application

$$\Gamma \vdash (V'[x := R'])(W'R'). \qquad \square$$

Lemma 6.3 *If Δ is* SI*-legal for a given* SIS*, then there is a context Γ, legal for the* IS*, with the same specification, such that $\Delta =_\beta S(\Gamma)$.*

Proof By induction on the number n in Definition 3.5. If $n = 1$ then $\Delta =_\beta \{Xx\}$, where $\vdash_{\mathrm{SI}} sX$ and $x \notin FV(X)$.

By Lemma 6.2 and Subject Reduction for ISs there is an $X' =_\beta X$ such that $\vdash_{\mathrm{I}} sX'$. Thus $\Delta =_\beta \{X'x\}$ and as by a start rule $X'x \vdash_{\mathrm{I}} X'x$, we have that $X'x$ is legal.

If $n > 1$ we have $\Delta =_\beta \{X_1x_1, \ldots, X_nx_n\}$ where (i), (ii), and (iii) of Lemma 3.4 hold. It follows that $\{X_1x_1, \ldots, X_{n-1}x_{n-1}\}$ is also SI-legal and, by the induction hypothesis, that there is a legal context Γ^- such that $\{X_1x_1, \ldots, X_{n-1}x_{n-1}\} =_\beta S(\Gamma^-)$.

Now by Lemma 6.2, Lemma 3.4(ii) with $i = n$, and Subject Reduction we have $\Gamma^- \vdash_{\mathrm{I}} s_nX'_n$ where $X_n =_\beta X'_n$. So Γ^-, X'_nx_n is legal. As $\Delta =_\beta S(\Gamma^-, X'_nx_n)$ we have the required result. □

Theorem 6.4

$$\Delta \vdash_{\mathrm{SI}} X, \Delta \text{ SI-}legal \Rightarrow \exists\Gamma, Y[\Delta =_\beta S(\Gamma) \;\&\; X =_\beta Y \;\&\; \Gamma \vdash_I Y].$$

Proof This immediately follows from Lemmas 6.2 and 6.3. □

Theorem 6.5

$$\Gamma \vdash_{\mathrm{I}} X \Rightarrow \Gamma \vdash_{\mathrm{AI}} X.$$

Proof By induction on the derivation of $\Gamma \vdash_{\mathrm{I}} X$ (similar to the proof of Theorem 8.1 in [12]). All cases are obvious except where $\Gamma \vdash_{\mathrm{I}} X$ comes by the abstraction rule from

$$\Gamma, Yx \vdash UV, \text{ and} \tag{11}$$

$$\Gamma \vdash_{\mathrm{I}} s_3(GY(\lambda x.U)), \tag{12}$$

and $X \equiv GY(\lambda x.U)(\lambda x.V)$.

By (12) and the Sharpened Generation Lemma for ISs there is a triple $(s_1, s_2, s_3) \in \mathcal{R}$ such that

$$\Gamma \vdash_{\mathrm{I}} s_1Y, \tag{13}$$

and $\Gamma, Yx \vdash_{\mathrm{I}} s_2U$, where the derivation of (13) is shorter than that of $\Gamma \vdash_{\mathrm{I}} X$. If $U =_\beta R \in \mathcal{C}$, then the Sharpened Generation Lemma for ISs gives $s_2R \in [\mathcal{A}]$. Hence (+) holds for U and s_1. By the induction hypothesis applied to (11) and (13) we have

$$\Gamma, Yx \vdash_{\mathrm{AI}} UV \;\&\; \Gamma \vdash_{\mathrm{AI}} s_1Y$$

which, given (+), gives $\Gamma \vdash_{\mathrm{AI}} GY(\lambda x.U)(\lambda x.V)$. □

Theorem 6.6 *If* $(*)$ *holds,*

$$\Gamma \vdash_{\mathrm{AI}} X \Rightarrow \Gamma \vdash_{\mathrm{I}} X.$$

Proof Similar to the proof of Theorem 2.11(ii) in Section 5. We now use Lemma 4.22 instead of Lemma 4.11. In the last lines of the proof of Theorem 2.11(ii) we had that $s \in \mathcal{S}_{\mathrm{I}}$ because of $\Gamma \vdash A : s$, hence by (+), $s \in \mathcal{S}_1 = \{s \in \mathcal{S}_{\mathrm{I}} | \exists s_2, s_3[(s, s_2, s_3) \in \mathcal{R}]\}$. Now $\Gamma \vdash sU$ for some U, hence $s \in \mathcal{S}_{\mathrm{I}}$ by Theorem 4.28 and so again $s \in \mathcal{S}_1$.

Now we will prove that if $(*)$ holds then

$$\Delta \vdash_{\mathrm{SAI}} X, \Delta \text{ SAI-legal} \Rightarrow \exists\Gamma, Y[\Delta =_\beta S(\Gamma) \;\&\; X =_\beta Y \;\&\; \Gamma \vdash_{\mathrm{AI}} Y].$$

As in the proof of Theorem 6.4 this will follow from two lemmas. These are similar to Lemmas 6.2 and 6.3. □

Lemma 6.7 *If*

$$\Delta \vdash_{\mathrm{SAI}} X, \tag{14}$$

$\Delta =_\beta S(\Gamma)$*, where* Γ *is* AI-*legal and* $(*)$ *holds, then there exists a* $Y =_\beta X$ *such that*

$$\Gamma \vdash_{\mathrm{AI}} Y.$$

Proof As $(*)$ holds, by Theorems 6.5 and 6.6,

$$\Gamma \vdash_{\mathrm{I}} Z \Leftrightarrow \Gamma \vdash_{\mathrm{AI}} Z.$$

Hence IS properties such as Subject Reduction also hold for AISs. Now the proof of this lemma is the same as that of Lemma 6.2 except as follows.

Case abstraction $X \equiv GU(\lambda x.V)(\lambda x.W)$ and (14) is obtained from

$$\Delta, Ux \vdash_{\mathrm{SAI}} VW \ \ \& \ \Delta \vdash_{\mathrm{SAI}} sU$$

and

$$(+) \quad \exists s_2, s_3 \left[(s, s_2, s_3) \in \mathcal{R} \ \& \ \forall T (V =_\beta T \in \mathcal{C} \Rightarrow s_2 T \in [\mathcal{A}])\right]$$

where V is not β-equal to an abstract. The proof of the case is now similar to the proof of the abstraction case in Lemma 6.2. □

Lemma 6.8 *If* $(*)$ *holds, then*

$$\Delta \text{ SAI-}\textit{legal} \Rightarrow \Delta =_\beta S(\Gamma) \textit{ for some } \text{AI-}\textit{legal}\ \Gamma.$$

Proof As in Lemma 6.7, properties such as Subject Reduction hold for AISs. Therefore the proof of this lemma is the same as the proof of Lemma 6.3. □

Theorem 6.9 *If* $(*)$ *holds, then*

$$\Delta \vdash_{\mathrm{SAI}} X, \Delta \text{ SAI-}\textit{legal} \Rightarrow \exists \Gamma, Y \left[\Delta =_\beta S(\Gamma) \ \& \ X =_\beta Y \ \& \ \Gamma \vdash_{\mathrm{AI}} Y\right].$$

Proof Directly from Lemmas 6.7 and 6.8. □

Now we are going to prove

$$\Gamma \vdash_{\mathrm{AI}} X \Rightarrow S(\Gamma) \text{ is SAI-legal and } S(\Gamma) \vdash_{\mathrm{SAI}} X.$$

In the SAIS postulates of 3.4 we have the condition that in the abstraction rule Y is not β-equal to an abstract, we need to show this holds automatically for AISs.

Lemma 6.10 *If* $\Gamma \vdash_{\mathrm{AI}} X$ *then* $X \equiv YZ$ *where* Y *is not* β*-equal to an abstract.*

Proof Consider the AIS, ω, with the same $\mathcal{A}$ and $\mathcal{S}$ as the one considered here, but with $\mathcal{R}$ replaced by $\mathcal{S}^3$. Then $(*)$ holds for ω and

$$\Gamma \vdash^{\omega}_{\mathrm{AI}} X.$$

Now by Theorem 6.6,

$$\Gamma \vdash^{\omega}_{\mathrm{I}} X,$$

and by Lemma 4.23 we have that $X \equiv YZ$ where Y is not β-equal to an abstract. □

Theorem 6.11

$$\Gamma \vdash_{\mathrm{AI}} X \Rightarrow S(\Gamma) \textit{ is } \text{SAI-}\textit{legal and } S(\Gamma) \vdash_{\mathrm{SAI}} X.$$

Proof By induction on the derivation of $\Gamma \vdash_{\mathrm{AI}} X$. In the abstraction case we use Lemma 6.10. □

Theorem 6.12

$$\Delta \vdash_{\mathrm{SI}} X \ \&\ \Delta \ \textit{is}\ \mathrm{SI}\textit{-legal} \ \Rightarrow \Delta \vdash_{\mathrm{SAI}} X \ \&\ \Delta \ \textit{is}\ \mathrm{SAI}\textit{-legal}.$$

Proof By Theorems 6.4, 6.5, and 6.11 and the conversion rule for SAISs. □

Theorem 6.13 *If* $(*)$ *holds,*

$$\Delta \vdash_{\mathrm{SAI}} X \ \&\ \Delta \ \textit{is}\ \mathrm{SAI}\textit{-legal} \ \Rightarrow \Delta \vdash_{\mathrm{SI}} X \ \&\ \Delta \ \textit{is}\ \mathrm{SI}\textit{-legal}.$$

Proof By Theorems 6.9, 6.6, and 6.1 and the conversion rule for SISs. □

In Sections 7 and 8 we link PTS variants and our illative systems, and in Section 9 we give a link between PTSs and SAISs, the systems closest to the ICLs in the literature.

7 From Type Systems to Illative Systems

We define a translation [ ] of pseudoterms of PTSs into ICL pseudoterms in the following way.

Definition 7.1 ([ ])

$$\begin{array}{rcl} [x] & \equiv & x \\ [c] & \equiv & c \\ [MN] & \equiv & [M][N] \\ [\lambda x : A.M] & \equiv & \lambda x.[M] \\ [\Pi x : A.B] & \equiv & G[A](\lambda x.[B]) \\ [M : A] & \equiv & [A][M] \end{array}$$

If B is a set or sequence of judgments $M_i : A_i$, for some values of i, then $[B]$ is the set or sequence of judgments $[A_i][M_i]$, for the same values of i.

We need some lemmas about the translation [ ].

Lemma 7.2 *If* B *and* N *are pseudoterms,*

$$[B][x := [N]] \equiv [B[x := N]].$$

Proof By induction on B. □

Lemma 7.3

(i) *If* $M \twoheadrightarrow_\beta N$, *then* $[M] \twoheadrightarrow_\beta [N]$.
(ii) *If* $M =_\beta N$, *then* $[M] =_\beta [N]$.

Proof (i) It is sufficient to prove this for a single β-contraction. Let $(\lambda x : A.P)Q$ be the part of M that reduces to $P[x := Q]$ in N then

$$\begin{array}{rcl} [(\lambda x : A.P)Q] & \equiv & (\lambda x.[P])([Q]) \to_\beta [P][x := [Q]] \\ & \equiv & [P[x := Q]] \end{array}$$

by Lemma 7.2.

Any remaining parts of M are identical to the remaining parts of N and so any remaining parts of $[M]$ are identical to the remaining parts of $[N]$. Hence $[M] \to_\beta [N]$.

(ii) If $M =_\beta N$ there is a P such that $M \twoheadrightarrow_\beta P$ and $N \twoheadrightarrow_\beta P$. By (i), $[M] \twoheadrightarrow_\beta [P]$ and $[N] \twoheadrightarrow_\beta [P]$ so $[M] =_\beta [N]$. □

Lemma 7.4 *If A is a pseudoterm and $[A] \twoheadrightarrow_\beta X$, then there is a pseudoterm B such that $X \equiv [B]$ and $A \twoheadrightarrow_\beta B$.*

Proof It is enough to prove this for a single β-contraction. We do this by induction on A.

Case 1 $A \equiv \lambda x : C.M$, $[A] \equiv \lambda x.[M]$ and $X \equiv \lambda x.Y$ where $[M] \to_\beta Y$. By the induction hypothesis there is a pseudoterm N such that $Y \equiv [N]$ and $M \to_\beta N$. Then $B \equiv \lambda x : C.N$.

Case 2 $A \equiv \Pi x : D.C$, $[A] \equiv G[D](\lambda x.[C])$ and $X \equiv GU(\lambda x.V)$ where $[D] \to_\beta U$ and $V \equiv [C]$ or $[D] \equiv U$ and $[C] \to_\beta V$. By the induction hypothesis there is an E such that $D \to_\beta E$ and $U \equiv [E]$ or $C \to_\beta E$ and $V \equiv [E]$. Thus the lemma holds with $B \equiv \Pi x : E.C$ or $\Pi x : D.E$.

Case 3 $A \equiv CD$ and $X \equiv UV$ where $[C] \to_\beta U$ or $[D] \to_\beta V$. By the induction hypothesis there is an E such that $D \to_\beta E$ and $[E] \equiv V$ or $C \to_\beta E$ and $[E] \equiv U$. Thus $B \equiv CE$ or ED.

Case 4 $A \equiv (\lambda x : C.M)N$ where $X \equiv [M][x := [N]]$. Then $X \equiv [M[x := N]]$ by Lemma 7.2, so $B \equiv M[x := N]$. □

Note that in this proof we use that G is primitive, hence not an abstract.

We can now prove that [ ] translations of valid (S)(A)PTS judgments are valid in the corresponding ICL.

Theorem 7.5 *Let X denote $\varnothing$, S, or A, then*

$$\Gamma \vdash_X M : A \Rightarrow [\Gamma] \vdash_{XI} [A][M].$$

Proof By straightforward induction on the derivation of $\Gamma \vdash_X M : A$. □

Theorem 7.6 *If $(*)$ holds,*

$$\Delta \vdash_{\mathrm{SA}} M : A \;\&\; \Delta \textit{ is } \mathrm{SA}\textit{-legal} \Rightarrow [\Delta] \vdash_{\mathrm{SAI}} [A][M] \;\&\; [\Delta] \textit{ is } \mathrm{SAI}\textit{-legal}.$$

Proof By Theorems 2.12(ii), 7.5, 6.11, and SAI conversion applied to the legality of Δ and to $\Delta \vdash_{\mathrm{SA}} M : A$. □

8 From Illative Systems to Type Systems

Definition 8.1 (of $\sim$) If A_1 and A_2 are pseudoterms,

$$A_1 =_\beta A_2 \Rightarrow A_1 \sim A_2,$$
$$A_i =_\beta \Pi x_1 : B_1 \ldots \Pi x_n : B_n.s_i (i = 1, 2) \Rightarrow A_1 \sim A_2.$$

Lemma 8.2

$$\left.\begin{array}{l}\Gamma \vdash P_1 : B_1,\;\; \Gamma \vdash P_2 : B_2 \\ B_1 \sim B_2, [P_1] \equiv [P_2], P_1, P_2 \textit{ in normal form}\end{array}\right\} \Rightarrow P_1 \equiv P_2.$$

Proof By induction on the structure of P_1.

Cases 1 and 2 $P_1 \equiv x$ and $P_1 \equiv c$ are trivially okay.

Case 3 $P_1 \equiv \lambda x : A_1.M_1$.

$[P_1] \equiv [P_2]$ so $P_2 \equiv \lambda x : A_2.M_2$. By the Generation Lemma for PTSs we get

$$B_i =_\beta \Pi x : A_i.C_i, \qquad \Delta, x : A_i \vdash M_i : C_i \qquad (i = 1, 2).$$

$B_1 \sim B_2$ so $A_1 =_\beta A_2$, hence $A_1 \equiv A_2$ because P_1 and P_2 are in normal form. We have

$$\Delta, x : A_1 \vdash M_1 : C_1, \Delta, x : A_1 \vdash M_2 : C_2, [M_1] \equiv [M_2],\ C_1 \sim C_2,$$

so $M_1 \equiv M_2$ by the induction hypothesis; hence $P_1 \equiv P_2$.

Case 4 $P_1 \equiv \Pi x : A_1.C_1$. $[P_1] \equiv [P_2]$ so $P_2 \equiv \Pi x : A_2.C_2, [A_1] \equiv [A_2]$, $[C_1] \equiv [C_2]$. We get

$$\Delta \vdash A_1 : s_1, \Delta \vdash A_2 : s_2, [A_1] \equiv [A_2] \Rightarrow A_1 \equiv A_2,$$

hence

$$\Delta, x : A_1 \vdash C_1 : s_1', \quad \Delta, x : A_1 \vdash C_2 : s_2', \quad [C_1] \equiv [C_2] \ \Rightarrow\ C_1 \equiv C_2.$$

So $P_1 \equiv P_2$.

Case 5 $P_1 \equiv M_1 N_1$. $[P_1] \equiv [P_2]$ so $P_2 \equiv M_2 N_2, [M_1] \equiv [M_2], [N_1] \equiv [N_2]$. We distinguish five cases:

$$M_1 \equiv c, \quad M_1 \equiv \Pi x : D_1.E_1, \quad M_1 \equiv \lambda x : D_1.E_1, \quad M_1 \equiv y, \quad M_1 \equiv D_1 E_1.$$

The first two cases cannot occur by the Generation Lemma for PTSs. Also the third case is not applicable because P_1 is in normal form. In the last case there are again two possibilities: $D_1 \equiv y$, $D_1 \equiv F_1 G_1$. It turns out that the last two cases reduce to the one case

$$P_1 \equiv y N_1 \ldots N_n, P_2 \equiv y L_1 \ldots L_n, [N_i] \equiv [L_i], n > 0.$$

As $n > 0$ we have

$$\Delta \vdash y : \Pi x_1 : F_1.F_2, \Delta \vdash N_1 : F_1, \Delta \vdash y N_1 : F_2[x_1 := N_1],$$

$$\Delta \vdash y : \Pi x_1 : H_1.H_2,\ \Delta \vdash L_1 : H_1,\ \Delta \vdash y L_1 : H_2[x_1 := L_1].$$

The Generation Lemma for PTSs yields $\Pi x_1 : F_1.F_2 =_\beta \Pi x_1 : H_1.H_2$ and so $F_1 =_\beta H_1$ and $\Delta \vdash L_1 : F_1$; hence we have by the induction hypothesis $N_1 \equiv L_1$. So $F_2[x_1 := N_1] =_\beta H_2[x_1 := L_1]$. Now suppose $n > 1$. Then

$$\Delta \vdash y N_1 : \Pi x_2 : F_3.F_4 =_\beta F_2[x_1 := N_1], \Delta \vdash N_2 : F_3, \Delta \vdash y N_1 N_2 : F_4[x_2 := N_2],$$

$$\Delta \vdash y N_1 : \Pi x_2 : H_3.H_4 =_\beta H_2[x_1 := L_1], \Delta \vdash L_2 : H_3, \Delta \vdash y N_1 L_2 : H_4[x_2 := L_2].$$

Now $\Pi x_2 : F_3.F_4 =_\beta \Pi x_2 : H_3.H_4$. So $N_2 \equiv L_2$ and hence $F_4[x_2 := N_2] =_\beta H_4[x_2 := L_2]$. Continuing in this way we get finally $N_i \equiv L_i$ for all i. □

We need a version of Lemma 8.2 with $\equiv$ replaced by $=_\beta$. We can only prove that for PTSs where each legal term has a normal form.

From now on we restrict ourselves to normalizing PTSs , that is, PTSs such that each legal term has a normal form.

Lemma 8.3 *If P_1 and P_2 are pseudo* SA-*terms in normal form then*

$$[P_1] =_\beta [P_2] \Rightarrow [P_1] \equiv [P_2].$$

Proof By Church-Rosser and Lemma 7.4. □

Lemma 8.4

$$\left.\begin{array}{ll} \Gamma \vdash P_1 : B_1, & \Gamma \vdash P_2 : B_2 \\ B_1 \sim B_2,\ [P_1] & =_\beta [P_2] \end{array}\right\} \Rightarrow P_1 =_\beta P_2.$$

Proof Let Q_i be the normal form of P_i. Then $[Q_1] =_\beta [Q_2]$ and by Lemmas 8.2 and 8.3, we get $Q_1 \equiv Q_2$ and so $P_1 =_\beta P_2$. □

Lemma 8.5 *If* $\Gamma_1 \vdash M : A, \Gamma_2 \vdash N : B$ *and* $[\Gamma_1] =_\beta [\Gamma_2]$*, then* $\Gamma_1 =_\beta \Gamma_2$.

Proof By induction on the length of Γ_1. Let

$$\Gamma_1 \equiv x_1 : A_1, \ldots, x_n : A_n, \text{ and}$$
$$\Gamma_2 \equiv x_1 : B_1, \ldots, x_n : B_n.$$

By Lemma 4.4, we have $s_1, s_2 \in \mathcal{S}$ such that

$$x_1 : A_1, \ldots, x_{n-1} : A_{n-1} \vdash A_n : s_1$$

and

$$x_1 : B_1, \ldots, x_{n-1} : B_{n-1} \vdash B_n : s_2.$$

So by the induction hypothesis, if $[\Gamma_1] =_\beta [\Gamma_2]$,

$$x_1 : A_1, \ldots, x_{n-1} : A_{n-1} =_\beta x_1 : B_1, \ldots, x_{n-1} : B_{n-1}.$$

By Lemma 4.10,

$$x_1 : A_1 \ldots x_{n-1} : A_{n-1} \vdash B_n : s_2,$$

and by Lemma 8.4, $A_n =_\beta B_n$, that is, $\Gamma_1 =_\beta \Gamma_2$. □

Theorem 8.6 *If in the* IS *corresponding to a normalizing* PTS,

$$\Gamma \vdash_I UV, \tag{1}$$

then there exist Γ_1, A, M *such that* $\Gamma \equiv [\Gamma_1]$, $U \equiv [A]$, $V \equiv [M]$*, and*

$$\Gamma_1 \vdash M : A. \tag{2}$$

Proof By induction on the derivation of (1).

Case 1 (Axiom) $\vdash_I sc$. This case is trivially okay.

Case 2 (Start)

$$\frac{\Gamma^- \vdash_I sU}{\Gamma^-, Ux \vdash_I Ux}$$

where $\Gamma \equiv \Gamma^-, Ux$ and $V \equiv x$. By the induction hypothesis

$$\Gamma_1^- \vdash A : B$$

where

$$\Gamma^- \equiv [\Gamma_1^-], s \equiv [B] \text{ and } U \equiv [A], \text{ hence } B \equiv s, \text{ so } \Gamma_1^- \vdash A : s.$$

Now by start $\Gamma_1^-, x : A \vdash x : A$ which is (2).

Case 3 (Weakening)

$$\frac{\Gamma^- \vdash_I UV \quad \Gamma^- \vdash_I sY}{\Gamma^-, Yx \vdash_I UV}$$

where $\Gamma \equiv \Gamma^-, Yx$. By the induction hypothesis we have

$$\Gamma_2 \vdash M : A, \Gamma_3 \vdash C : B$$

where

$$\Gamma^- \equiv [\Gamma_2] \equiv [\Gamma_3], V \equiv [M], U \equiv [A], Y \equiv [C] \text{ and, as above, } B \equiv s.$$

By Lemmas 8.5 and 4.10, $\Gamma_2 \vdash C : s$ and by a weakening rule we have (2) with $\Gamma_1 \equiv \Gamma_2, x : C$.

Case 4 (Conversion)

$$\frac{\Gamma \vdash_{\mathrm{I}} UV \quad \Gamma \vdash_{\mathrm{I}} sY \quad Y =_\beta U}{\Gamma \vdash_{\mathrm{I}} YV}.$$

By the induction hypothesis we get similarly to above,

$$\Gamma_1 \vdash M : B, \Gamma_1 \vdash A : s \text{ where } \Gamma \equiv [\Gamma_1], U \equiv [B], V \equiv [M], \text{and } Y \equiv [A].$$

By Lemmas 4.8, 7.4, and 8.4, we get $A =_\beta B$. Hence by conversion $\Gamma_1 \vdash M : A$ which is (2).

Case 5 (Application)

$$\frac{\Gamma \vdash_{\mathrm{I}} GU(\lambda x.V)T \quad \Gamma \vdash_{\mathrm{I}} UR}{\Gamma \vdash_{\mathrm{I}} (V[x := R])\,(TR)}.$$

We get similarly to above

$$\Gamma_1 \vdash N : B, \qquad \Gamma_1 \vdash P : C$$

where

$$\Gamma \equiv [\Gamma_1], T \equiv [N],\ GU(\lambda x.V) \equiv [B],\ U \equiv [C],\ \text{and } R \equiv [P],$$

$$[B] \equiv GU(\lambda x.V) \text{ so } B \equiv \Pi x : E.F \text{ where } U \equiv [E] \text{ and } V \equiv [F].$$

By Lemmas 4.8 and 4.7(iii), $\Gamma_1 \vdash E : s$, for some s, so we have $\Gamma_1 \vdash P : C,\ C =_\beta E$ (by 4.8 and 8.4), $\Gamma_1 \vdash E : s$.

Hence by conversion

$$\Gamma_1 \vdash P : E.$$

Hence $\Gamma_1 \vdash NP : F[x := P]$ which is (2).

Case 6 (Abstraction)

$$\frac{\Gamma, Yx \vdash_I UV \quad \Gamma \vdash_I s(GY(\lambda x.U))}{\Gamma \vdash_I GY(\lambda x.U)(\lambda x.V)}.$$

By the induction hypothesis we get, similarly to above,

$$\Gamma_1, x : B \vdash N : C, \quad \Gamma_1 \vdash \Pi x : D.E : s$$

where

$$\Gamma \equiv [\Gamma_1], Y \equiv [B], U \equiv [C], V \equiv [N], Y \equiv [D], \text{and } U \equiv [E].$$

By Lemmas 4.8 and 8.4 we get $B =_\beta D$ and $C =_\beta E$.

By Lemma 4.10 and the Generation Lemma we get

$$\Gamma_1, x : D \vdash N : C.$$

Now $C =_\beta E$ and $\Gamma_1, x : D \vdash E : s'$ for some s'. So by conversion

$$\Gamma_1, x : D \vdash N : E.$$

Hence by abstraction

$$\Gamma_1 \vdash \lambda x : D.N : \Pi x : D.E$$

which is (2).

Case 7 (Product)

$$\frac{\Gamma, Yx \vdash_I s_2 Z \quad \Gamma \vdash_I s_1 Y}{\Gamma \vdash_I s_3(GY(\lambda x.Z))}$$

where $(s_1, s_2, s_3) \in \mathcal{R}$. We get

$$\Gamma_1, x : B \vdash C : s_2 \quad \Gamma_1 \vdash D : s_1$$

where

$$\Gamma \equiv [\Gamma_1], Y \equiv [B], Z \equiv [C], \text{ and } Y \equiv [D].$$

By Lemma 8.4 we get $B =_\beta D$, hence $\Gamma_1, x : D \vdash C : s_2$ by Lemma 4.10.

So by product,

$$\Gamma_1 \vdash \Pi x : D.C : s_3,$$

which is (2). □

Theorem 8.7 *For* SIS*s and* SPTS*s such that the corresponding* PTS *is normalizing, if* Δ *is* SI-*legal and*

$$\Delta \vdash_{\mathrm{SI}} X,$$

then there exist Δ_1, *A, and M such that*

$$\Delta =_\beta [\Delta_1], X =_\beta [A][M], \Delta_1 \text{ S-}\textit{legal, and } \Delta_1 \vdash_{\mathrm{S}} M : A.$$

Proof From Theorems 6.4, 8.6, and 2.10(i). □

Theorem 8.8 *For* AIS*s and* APTS*s such that the corresponding* PTS *is normalizing and satisfies* $(*)$, *if*

$$\Gamma \vdash_{\mathrm{AI}} X,$$

then there exist Γ_1, *A, and M such that*

$$\Gamma \equiv [\Gamma_1], X \equiv [A][M], \textit{ and } \Gamma_1 \vdash_{\mathrm{A}} M : A.$$

Proof From Theorems 6.6, 8.6, and 2.11(i). □

Theorem 8.9 *For* SAIS*s and* SAPTS*s such that the corresponding* PTS *is normalizing and satisfies* $(*)$, *if* Δ *is* SAI-*legal and*

$$\Delta \vdash_{\mathrm{SAI}} X,$$

then there exist Δ_1, *A, and M such that*

$$\Delta =_\beta [\Delta_1], X =_\beta [A][M], \Delta_1 \text{ SA-}\textit{legal, and } \Delta_1 \vdash_{\mathrm{SA}} M : A.$$

Proof From Theorems 6.9, 8.8, and 2.12(i). □

9 Linking PTSs and SAISs

We are now able to link PTSs to SAISs, the systems closest to the illative systems in the literature.

Theorem 9.1

$$\Gamma \vdash M : A \Rightarrow [\Gamma] \vdash_{\mathrm{SAI}} [A][M].$$

Proof By Theorems 7.5, 6.5, and 6.11. □

Theorem 9.2

$$\Delta \vdash_{\mathrm{SAI}} X \Rightarrow \exists \Gamma, M, A\ [S(\Gamma) =_\beta \Delta, [A][M] =_\beta X \ \&\ \Gamma \vdash M : A],$$

provided Δ *is* SAI-*legal, the* PTS *is normalizing, and* $(*)$ *holds.*

Proof By Theorems 6.9, 6.6, and 8.6. □

10 PTSs and ICLs in the Literature

Illative systems of combinatory logic such as these of Bunder [11], [3], [4], and [7], the later "Frege Structures" of Aczel [1], and the version of the Calculus of Constructions in Coquand [14] and Seldin [17] are slightly more general than the SAIs that we have developed here in that in

$$\Delta, Xx \vdash YZ$$

in the abstraction rule, Y may be an abstract and in that (+) need not hold. Some have additional definitions and postulates such as conversion with $\beta\eta$-equality. Still by Theorem 9.1, the translation of any valid PTS judgment is valid in these illative systems. By Theorem 9.2, a subclass of the theorems of these illative systems can be translated back into PTSs.

It was thought that setting up the above two links between PTSs and SAISs would allow a transfer of properties from one to the other. We will examine the most important such property, that of consistency.

10.1 SAIS consistency The original illative systems of Church and Curry were inconsistent in the strong sense that every term (including an arbitrary variable or λ-term) was provable. Some later systems that included a class of propositions $\mathbf{H}$ were inconsistent in the weaker sense (see Bunder [5], [6], and [9] and Bunder and Meyer [10]) that all propositions were provable. This was expressed as $\vdash \Xi\mathbf{HI}$, which can be translated into $\vdash G\mathbf{H}(\lambda xy.y)(\lambda x.x)$ and, by Definition 7.1, with $*$ for $\mathbf{H}$, into $\vdash (\lambda x : *.x) : (\Pi x : *.\lambda y.y)$.

By the Sharpened Generation Lemma (4.7) this is not a valid judgment of a PTS, so by Theorem 9.2, SAISs are consistent in the strong sense that not all propositions are provable, if the corresponding PTS is normalizing and satisfies $(*)$.

In fact $(\Pi x : *.\lambda y.y)$ is, by the Correctness of Types Lemma (4.8), not even a possible type, so it seems that $\Xi\mathbf{HI}$ cannot be represented in a SAIS given normalization and $(*)$.

In many ICLs in the literature, however, it is important to have $\Xi\mathbf{HI}$ as a proposition so that negation can be defined by $\sim X \equiv X \supset \Xi\mathbf{HI}$. Also, in these, the properties of intuitionistic implication and negation are derived from the postulates for Ξ (or G) using either the definition $\mathbf{H} \equiv \lambda x.\mathbf{L}(\lambda y.x)$ or $\mathbf{L} \equiv \mathbf{F}U\mathbf{H}$ for some U and the axiom $\vdash \mathbf{LH}$. $\vdash \mathbf{LH}$ is the counterpart to $\vdash * : \square$, a standard PTS axiom, but to have sorts defined in terms of other sorts and having types that are abstracts is not possible in PTSs or in SAISs. Hence a gap remains between SAISs and the illative systems in the literature.

10.2 PTS consistency A PTS is inconsistent if, for some M,

$$\vdash M : (\Pi x : *.x),$$

that is, if M is a proof that every proposition (element of $*$) is a theorem.

If this were valid we would have, in the corresponding SAIS,

$$\vdash G * (\lambda x.x)[M]$$

or

$$*x \vdash x([M]x).$$

If $*$ is interpreted as $\mathbf{H}$, the class of propositions, this is unprovable (and in fact ill formed as was the translation of SAIS inconsistency into PTSs). However, if $*$ is interpreted instead as a class of sets and the term $[M]$ as a choice function, the result is in fact true!

10.3 Why the mismatch? The reason for the mismatch is, of course, that in a PTS only the type is considered as a proposition of predicate calculus, whereas in illative systems the translation of the term and the type, that is, a whole statement is considered as a proposition. Despite this we have seen that the postulates of PTSs and (S)(A)ISs are remarkably similar and in fact equivalent, modulo legality, β-equality, and $(*)$.

References

[1] Aczel, P., "Frege structures and the notions of proposition, truth and set," pp. 31–59 in *The Kleene Symposium (University of Wisconsin, Madison, 1978)*, vol. 101 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1980. Zbl 0462.03002. MR 82e:03045. 181, 203

[2] Barendregt, H. P., "Lambda calculi with types," pp. 117–309 in *Handbook of Logic in Computer Science*, vol. 2 of *Oxford Science Publications*, Oxford University Press, New York, 1992. MR 97m:03026. 188

[3] Bunder, M. W., "A deduction theorem for restricted generality," *Notre Dame Journal of Formal Logic*, vol. 14 (1973), pp. 341–46. Zbl 0197.28204. MR 48:5828. 181, 182, 188, 203

[4] Bunder, M. W., "Propositional and predicate calculuses based on combinatory logic," *Notre Dame Journal of Formal Logic*, vol. 15 (1974), pp. 25–34. Zbl 0272.02045. MR 49:8818. 182, 203

[5] Bunder, M. W., "Consistency notions in illative combinatory logic," *The Journal of Symbolic Logic*, vol. 42 (1977), pp. 527–29. Zbl 0382.03013. MR 58:16176. 203

[6] Bunder, M. W., "Equality in $\mathcal{F}_{21}^*$ with restricted subjects," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 24 (1978), pp. 125–27. Zbl 0422.03002. MR 80e:03011. 203

[7] Bunder, M. W., "On the equivalence of systems of rules and systems of axioms in illative combinatory logic," *Notre Dame Journal of Formal Logic*, vol. 20 (1979), pp. 603–08. Zbl 0349.02020. MR 80h:03019. 203

[8] Bunder, M. W., "Possible forms of evaluation or reduction in Martin-Löf type theory," *Theoretical Computer Science*, vol. 41 (1985), pp. 113–20. Zbl 0612.03007. MR 88a:03150. 181

[9] Bunder, M. W., "Some consistency proofs and a characterization of inconsistency proofs in illative combinatory logic," *The Journal of Symbolic Logic*, vol. 52 (1987), pp. 89–110. Zbl 0614.03015. MR 88k:03026. 203

[10] Bunder, M. W., and R. K. Meyer, "On the inconsistency of systems similar to $\mathcal{F}_{21}^*$," *The Journal of Symbolic Logic*, vol. 43 (1978), pp. 1–2. Zbl 0394.03022. MR 80a:03002a. 203

[11] Bunder, M. W. V., *Set Theory Based on Combinatory Logic*, Universiteit van Amsterdam, Amsterdam, 1969. Doctoral dissertation, University of Amsterdam. MR 49:8817. 181, 188, 203

[12] Bunder, M., and W. Dekkers, "Pure type systems with more liberal rules," *The Journal of Symbolic Logic*, vol. 66 (2001), pp. 1561–80. Zbl 0997.03014. MR 2002m:03016. 182, 185, 186, 188, 193, 195

[13] Bunder, M. W., "The naturalness of illative combinatory logic as a basis for mathematics," pp. 55–64 in *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, Academic Press, London, 1980. MR 82a:03015. 182

[14] Coquand, T., "Metamathematical investigations of a calculus of constructions," pp. 91–122 in *Logic and Computer Science*, edited by P. Odifreddi, Academic Press, London, 1990. 203

[15] Curry, H., J. Hindley, and J. Seldin, *Combinatory Logic*, vol. 2, Universiteit van Amsterdam, Amsterdam, 1972. Zbl 0242.02029. 182, 187

[16] Martin-Löf, P., "Constructive mathematics and computer programming," Technical Report Report 11, University of Stockholm, 1979. Zbl 0443.68039. 181

[17] Seldin, J. P., "On the proof theory of Coquand's calculus of constructions," *Annals of Pure and Applied Logic*, vol. 83 (1997), pp. 23–101. Zbl 0873.03048. MR 98e:03019. 203

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